Transformation of Sturm-Liouville problems with decreasing affine boundary conditions *

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Abstract

We consider Sturm-Liouville boundary value problems on the interval $[0,1]$ of the form $-y'' + qy = \lambda y$ with boundary conditions $y'(0)\sin\alpha = y(0)\cos\alpha$ and $y'(1) = (a\lambda + b)y(1)$, where $a < 0$. We show that via multiple Crum-Darboux transformations, this boundary value problem can be "almost" isospectrally transformed to a boundary value problem of the same form, but with the boundary condition at $x = 1$ replaced by $y'(1)\sin\beta = y(1)\cos\beta$, for some β.

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1 Introduction

Our aim is to transform, "almost" isospectrally, a Sturm-Liouville equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad 0 \le x \le 1 \tag{1.1}$$

with boundary conditions

$$Y(0) = \cot\alpha, \qquad 0 \le \alpha < \pi \tag{1.2}$$

$$Y(1) = a\lambda + b, \tag{1.3}$$

where $a < 0$, into a "standard" Sturm-Liouville problem. By "almost" we mean that at most two eigenvalues will change, and by "standard" we mean a problem where the differential equation is regular and the boundary conditions are independent of λ. We shall consistently use upper case Roman letters to denote logarithmic derivatives, so Y means y'/y in (1.2), (1.3). We assume that q is real and integrable on $[0,1]$, and if $\alpha = 0$ then (1.2) is interpreted as $y(0) = 0$. The decision to keep (1.2) independent of λ is for simplicity of presentation – cf. [5] for analogous questions with both boundary conditions λ-dependent.

Sturm-Liouville problems with λ-dependent boundary conditions of the form

$$Y(1) = f(\lambda)$$

have been studied a good deal from the viewpoints of both theory and applications. Most applications are to affine conditions like (1.3) – [10] and [18] have extensive reference lists, but see, e.g., [3] for square root dependence and also [13] for the bilinear case

$$f(\lambda) = \frac{a\lambda + b}{c\lambda + d}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0. \tag{1.4}$$

Theoretical investigations involving Herglotz-Nevanlina functions f can be found in [17], rational f in, e.g., [16], and a combination of these properties was considered in [7]. More general λ-dependence, where f is a ratio of holomorphic functions, was studied in, e.g., [12]. We hope to use the material here as a foundation for treating some of the above cases, and also in the study of inverse spectral problems, cf. [8].

The transformation we seek was carried out for the case $a > 0$ in [6], and we now briefly describe some of the ideas involved for the simplest (non-Dirichlet) case, $\alpha > 0$. We start with a "base function" z, i.e., a non-vanishing solution of (1.1) for some fixed λ. Then $Z(= z'/z)$ can be used to transform y to $\hat{y} = y' - Zy$ and q to $\hat{q} = q - 2Z'$ in (1.1). Equivalent expressions were given by Darboux [14, p. 132] and we shall call this a Darboux transformation. Darboux did not consider boundary conditions, but if we require z to obey (1.2)-(1.3) then $\hat{y}$ satisfies boundary conditions independent of λ.

In [6], z was chosen as an eigenfunction of (1.1)-(1.3), and to be sure that an eigenfunction of one sign exists, one needs some oscillation theory, which is conveniently carried out via the Prüfer angle θ. Indeed the eigenvalues $\lambda_0, \lambda_1, ...$ are given by the abscissae at the intersections of the line (1.3) with the graph of

$$Y(1) = \cot \theta(1, \lambda).$$

The latter has countably many branches $\mathcal{B}_0, \mathcal{B}_1, ...$, points on $\mathcal{B}_k$ corresponding to solutions, y, of (1.1)-(1.2) with k zeros on $(0, 1)$. Since (1.3) (for $a > 0$) intersects $\mathcal{B}_0$ (and indeed each $\mathcal{B}_k$) precisely once, an eigenfunction z exists as required.

In our case $a < 0$, however, (1.3) need not intersect $\mathcal{B}_0$, and even if it does so, there will be two intersections (counted algebraically). Roughly, one net "extra" eigenvalue has been introduced, compared with the case $a > 0$. When (1.3) does not intersect $\mathcal{B}_0$, the "extra" eigenvalue is paired with the missing one from $\mathcal{B}_0$ to give either two extra eigenvalues (counted algebraically) on some further branch $\mathcal{B}_k$, or else one nonreal conjugate pair. The various possibilities (and their connections with algebraic multiplicities of eigenvalues) are analysed in Section 2.

Continuing with the case $a < 0 < \alpha$, we find that a Darboux transformation still reduces (1.1)-(1.3) to a "standard" problem if (1.3) intersects $\mathcal{B}_0$. This case, where the "extra" eigenvalue is "removed" from $\mathcal{B}_0$, will be detailed in Section 4. In the case where all eigenvalues are real, it turns out that an extension of Darboux's transformation involving two base functions is needed to remove the extra eigenvalue from a further branch $\mathcal{B}_k$ for $k > 0$. This contrasts with the case of (1.4), where a single base function suffices for such removal [6].

A Darboux-type transformation with multiple base functions (whose Wronskian replaces z used previously) was described by Crum [11], and we shall refer to our version noted above as a (double) Crum transformation. (The names of Darboux and/or Crum are associated with such transformations in much of the literature.) Actually Crum used the first n eigenfunctions for his base functions, corresponding to n iterated Darboux (whom he did not reference) transformations. Our version is closer to that of Adler, who allowed two eigenfunctions with oscillation counts differing by one [1, Lemma 1], but both Crum and Adler produced singular transformed problems. To achieve regularity, we instead use two base functions which need not be eigenfunctions, but have the same oscillation count. The background for all the transformations we need (which for nonreal eigenvalues use up to four base functions) is presented in Section 3.

The Dirichlet case $\alpha = 0$ is more complicated than $\alpha > 0$, and for example a double Crum transformation may be needed even when (1.3) intersects $\mathcal{B}_0$, while two such transformations in tandem are required when (1.1)-(1.3) has a triple eigenvalue. All cases requiring double Crum transformations are covered in Section 5. Finally, nonreal eigenvalues of (1.1)-(1.3) are treated in Section 6. When $\alpha > 0$, a triple Crum transformation produces a standard problem, but when $\alpha = 0$, one needs a quadruple Crum transformation followed by a single one (i.e., of Darboux type).

2 Preliminaries

We shall rely on Prüfer theory associated with (1.1), (1.2). If $y(x,\lambda)$ is a solution of (1.1) and (1.2) then we put

$$y = \rho \sin\theta, \qquad y' = \rho\cos\theta,$$

where θ is the Prüfer angle associated with (1.1) and (1.2). Differentiating, we see that θ obeys the first order initial value problem

$$\theta' = \cos^2\theta + (\lambda - q)\sin^2\theta, \qquad \theta(0,\lambda) = \alpha.$$

Atkinson [2] provides a comprehensive account of this theory but it suffices here for us to note that $\theta(1,\lambda)$ is increasing in λ, $\theta(x,\lambda) \to 0$ as $\lambda \to -\infty$ and $\theta(x,\lambda) \to \infty$ as $\lambda \to \infty$ for each $x \in (0,1]$. The graph of $\cot\theta(1,\lambda)$ has branches $\mathcal{B}_0$, $\mathcal{B}_1$, $\cdots$ corresponding to λ-intervals $(-\infty, \lambda_0^D]$, $(\lambda_0^D, \lambda_1^D]$, $\cdots$ where the λ_n^D, $n \geq 0$, are the eigenvalues for (1.1), (1.2) with the Dirichlet condition $y(1) = 0$. Further, $\cot\theta(1,\lambda)$ is decreasing on each branch and $\cot\theta(1,\lambda) \to \pm\infty$ as $\lambda \to \lambda_n^D\pm$.

Real eigenvalues for (1.1)-(1.3) occur at λ values for which

$$\cot\theta(1,\lambda) = a\lambda + b.$$

A real eigenvalue $\hat{\lambda}$ is said to have algebraic multiplicity $k \geq 1$ if, for $ly = -y'' + qy$, there is a chain of functions $y^{[0]}$, $\cdots$, $y^{[k-1]}$ with $(l-\hat{\lambda})y^{[0]} = 0$, $(l-\hat{\lambda})(y^{[j]}) = y^{[j-1]}$ and $y^{[j]}$ satisfy the boundary conditions (1.2) (as this boundary condition is independent of λ) and

$$y^{[j]\prime}(1) = y^{[j]}(1)(a\hat{\lambda} + b) + a y^{[j-1]}$$

for each $1 \leq j \leq k-1$, and the chain cannot be extended to length $k+1$. Here $y^{[0]}$ is an eigenfunction for $\hat{\lambda}$ and $y^{[1]}$, $\cdots$, $y^{[k-1]}$ are the associated functions – see [15, pages 16-20] for more details. The algebraic multiplicity of an eigenvalue $\hat{\lambda}$ is k if the functions $\cot\theta(1,\lambda)$ and $a\lambda + b$ and their λ-derivatives of order 1, 2, $\cdots$, $k-1$ (but not k) agree at $\hat{\lambda}$ – see [9, Lemma 2.1, Theorem 3.1] and [15, pp. 16-20].

The following theorem on existence, multiplicity and asymptotics for eigenvalues of (1.1)-(1.3) will be a key tool in this work. From now on for simplicity we shall refer to an eigenvalue as "belonging" to $\mathcal{B}_k$ if it is the abscissa of a point on $\mathcal{B}_k$.

Theorem 2.1 *The boundary value problem (1.1)-(1.3) has only point spectrum, which is countably infinite and accumulates at $+\infty$ and can thus be listed as λ_n, $n \geq 0$ with eigenvalues repeated according to algebraic multiplicity and ordered so as to have increasing real parts.*

(i) For large n

$$\lambda_n = \begin{cases} \left(n-\frac{1}{2}\right)^2\pi^2 + 2\cot\alpha + \frac{2}{a} + \int_0^1 q + o\left(\frac{1}{n}\right), & \alpha \neq 0 \\ n^2\pi^2 + \frac{2}{a} + \int_0^1 q + o\left(\frac{1}{n}\right), & \alpha = 0. \end{cases}$$

(ii) One of the following occurs:

(a) *All eigenvalues are real, there are algebraically two eigenvalues on the initial branch, $\mathcal{B}_0$, of the Prüfer graph and all other branches contain precisely one simple eigenvalue.*

(b) *All eigenvalues are real, $\mathcal{B}_0$ contains no eigenvalues but for some $k > 0$, $\mathcal{B}_k$ contains algebraically three eigenvalues and all other branches contain precisely one simple eigenvalue.*

(c) *There are two non-real eigenvalues appearing as a conjugate pair. $\mathcal{B}_0$ contains no eigenvalues and all other branches contain precisely one simple eigenvalue.*

Proof: The first sentence follows from [16], as does the fact that all but finitely many eigenvalues real and simple. The asymptotic development in (i) is derived in [9].

(ii) From [4, Theorem 2.4], either (a) occurs or there are no eigenvalues on $\mathcal{B}_0$.

In the latter case [4, Theorem 2.4] shows that if there are only real eigenvalues then each branch (other than $\mathcal{B}_0$) contains at least one eigenvalue and at most one branch may contain algebraically more than one (and up to three). The asymptotics for λ_n^D are well known (cf. [5]) and are as in (i) with n replaced by $n+1$. Moreover, since $a\lambda + b \to -\infty$ as $\lambda \to \infty$, we see that, for large n, $\lambda_{n+1} < \lambda_n^D$. Thus in this case there will be precisely three eigenvalues on some $\mathcal{B}_k$ for some $k > 0$, i.e., (b) holds.

Finally, in the case when complex eigenvalues are present [16] shows that they appear in conjugate pairs, so there are at least two complex eigenvalues. Moreover [4, Theorem 2.4] ensures that there is exactly one real eigenvalue from each branch $\mathcal{B}_k$ for $k > 0$. Using eigenvalue asymptotics as above, we see that there is exactly one conjugate pair of non-real eigenvalues, so (c) holds. ■

The above theorem is illustrated geometrically in Figure 1.

It is convenient to establish a short-hand for the cases of real eigenvalues (to be treated in Sections 4 and 5) given by the above theorem. (Non-real eigenvalues, i.e., case (c) of the theorem, will be covered in Section 6.) The letter D denotes a Dirichlet condition and N signifies a non-Dirichlet condition at $x = 0$. It is apparent that for cases (a) and (b) there is precisely one branch, say $\mathcal{B}_k$, containing algebraically more than one eigenvalue. Then we denote our problem as D_k or N_k depending on the boundary condition at $x = 0$. These cases will be subdivided according to eigenvalue multiplicity. For example $D_0(2)$

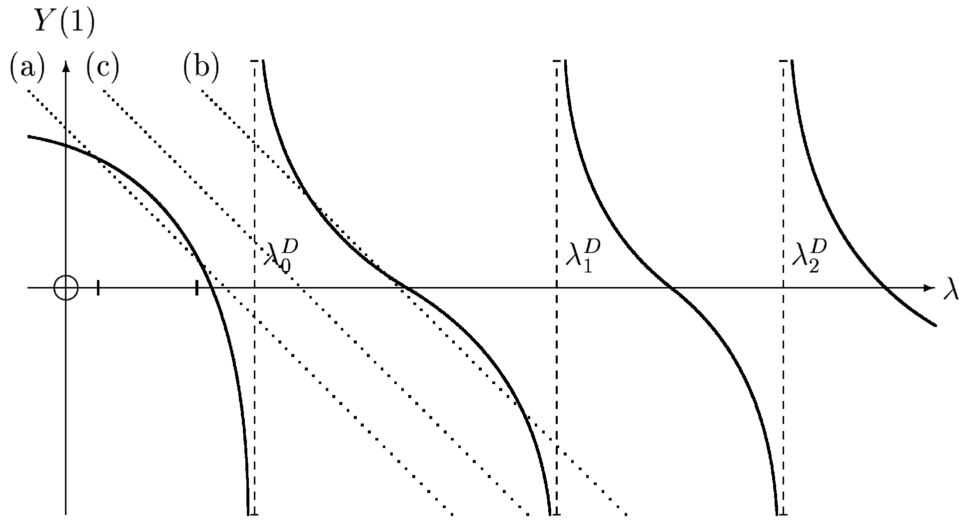

Figure 1: $\cot\theta(\lambda, 1)$

implies a double eigenvalue on $\mathcal{B}_0, N_k(2,1)$ implies a double eigenvalue followed by a simple one on $\mathcal{B}_k$ (necessarily $k > 0$), and so on.

3 Crum-type Transformations

In this section we introduce the (modified) Crum transformation and establish some of its essential properties. We begin by recalling the definition of Crum's transformation as given in [11].

Definition 3.1 *Suppose the functions $f_1, \cdots, f_k$ satisfy*

$$-f_j{}'' + qf_j = \lambda_j f_j, j = 1, \cdots, k. \tag{3.1}$$

Then the Wronskian is formally defined as the determinant

$$w(f_1, \cdots, f_k)(x) = \det\left[f_j^{(i-1)}(x)\right]_{i,j=1,\cdots,k}$$

in which all second and higher derivatives of f_j have been successively replaced by expressions involving f_j and f_j' for $j = 1, \cdots, k$.

We define the Crum transformation of a solution y of (1.1), with respect to the above base functions $f_1, \cdots, f_k$, by

$$e(x) = \frac{w(f_1, \cdots, f_k, y)(x)}{w(f_1, \cdots, f_k)(x)}, \tag{3.2}$$

where $w(f_1, \cdots, f_k)(x) \neq 0$ *on* $[0, 1]$.

In the case $k = 1$ we shall call this the Darboux transformation, cf. [14, p. 132]. Note, in the case of Sturm-Liouville boundary value problems with eigenparameter dependent boundary conditions of positive type, e.g. (1.3) with $a > 0$, (1.4) and in [7], [8], that it was enough to apply Darboux transformations with one base function, but here we shall need cases with up to four base functions.

Theorem 3.2 *Let* f_1, f_2, $\cdots$, f_n, f_{n+1} *be solutions of (1.1) with* λ *taking the values* μ_1, μ_2, $\cdots$, μ_{n+1} *respectively. Then for all* $x \in [0, 1]$ *where* $w(f_1, \cdots, f_n)(x) \neq 0$ *we have the following.*

(i) The function

$$\phi = \frac{w(f_1, \cdots, f_n, f_{n+1})}{w(f_1, \cdots, f_n)}$$

is a solution of the equation

$$-\phi'' + (q - 2W(f_1, \cdots, f_n)')\phi = \mu_{n+1}\phi.$$

(ii) If $n \geq 2$, *then the function*

$$\psi = \frac{w(f_1, \cdots, f_{n-1})}{w(f_1, \cdots, f_n)}$$

satisfies the equation

$$-\psi'' + (q - 2W(f_1, \cdots, f_n)')\psi = \mu_n\psi.$$

Proof: Part (i) is proved in [11], so we proceed to the proof of (ii). For convenience we write $w = w(f_1, \cdots, f_n)$, $v = w(f_1, \cdots, f_{n-1})$, $\phi = \frac{w}{v}$ and $\psi = \frac{v}{w}$. An easy calculation shows

$$\begin{aligned}
\phi'' &= \frac{vw'' - 2v'w' - v''w}{vw} + 2V^2\phi, \\
\psi'' &= \frac{wv'' - 2w'v' - w''v}{wv} + 2W^2\psi.
\end{aligned}$$

From (i), we have

$$\phi'' = (q - 2V' - \mu_n)\phi,$$

so

$$q - 2V' - \mu_n = \frac{vw'' - 2v'w' - v''w}{vw} + 2V^2$$

and

$$q - \mu_n = \frac{vw'' - 2v'w' + v''w}{vw}. \tag{3.3}$$

A similar consideration of ψ gives

$$q - 2W' - \mu_n = \frac{vw'' - 2v'w' - vw''}{vw} + 2W^2. \tag{3.4}$$

The result follows by combining (3.3) and (3.4). ■

The next theorem is used to ensure the non-vanishing of the Wronskian in later situations where two base functions are used; cf. [1] for a related result.

Theorem 3.3 *Let u, z be solutions of (1.1), (1.2) with λ replaced by μ and ξ, and α replaced by β and γ respectively. Suppose that u and z have the same number of zeros in $(0,1)$. If $\pi > \beta > \gamma \geq 0$ and $\mu > \xi$ then $w(u,z)$ is nonzero everywhere on $[0,1]$.*

Proof: We can assume without loss that u and z are normalized so that $u(0) = 1$, $u'(0) = \cot\beta$ and $z(0) = 1$, $z'(0) = \cot\gamma$ if $\gamma \neq 0$, $z(0) = 0$, $z'(0) = 1$ if $\gamma = 0$. First, we note that since

$$w(u,z)(0) = \begin{cases} \cot\gamma - \cot\beta, & \gamma \neq 0 \\ 1, & \gamma = 0 \end{cases}$$

we have that $w(u,z)(0) > 0$. Let the zeros of u in $(0,1)$ be $0 < a_1 < \cdots < a_m < 1$ and those of z be $0 < b_1 < \cdots < b_m < 1$. Sturm theory shows that $0 < a_1 < b_1 < \cdots < a_m < b_m < 1$.

Now

$$w'(u,z) = (\mu - \xi)\, uz$$

so the critical points of $w(u,z)$ occur at $x = a_j, b_j$, $1 \leq j \leq m$. From the interlacing of the a_j and the b_j we also see that

$$(-1)^j u'(a_j) > 0, \qquad (-1)^j u(b_j) > 0,$$
$$(-1)^j z'(b_j) > 0, \qquad (-1)^j z(a_j) < 0,$$

and hence $w(u,z) > 0$ at all of its critical points. Further

$$w''(u,z)(b_m) = (\mu - \xi) u(b_m) z'(b_m) > 0,$$

so the final critical point is a minimum. Thus $w(u,z) > 0$ on $[0,1]$. ■

The following theorem gives the analogue of Theorem 3.2 that will be used when transforming non-simple eigenvalues.

Theorem 3.4 *Let $l(y) = -y'' + qy$ and $y^{[j]}, j = 0, ..., k$, be solutions to the system*

$$l(y^{[0]}) = \hat{\lambda} y^{[0]} \tag{3.5}$$
$$l(y^{[j]}) = \hat{\lambda} y^{[j]} + y^{[j-1]}, \qquad j = 1, \cdots, k. \tag{3.6}$$

Suppose that $z_1, ...z_m$ are solutions of (1.1) with λ replaced by $\mu_1, ..., \mu_m$, $m \in \mathbb{N} \cup \{0\}$. Where $w(z_1, ..., z_m, y^{[0]})(x) \neq 0$, the functions

$$u^{[j-1]} = \frac{w(z_1, ..., z_m, y^{[0]}, y^{[j]})}{w(z_1, ..., z_m, y^{[0]})}, \qquad j = 1, \cdots, k$$

are solutions to the system (3.5)-(3.6) with k and q replaced by $k-1$ and $q - 2W(z_1, ..., z_m, y^{[0]})'$.

Proof: For each $\lambda \in \mathbb{C}$, let g_λ be the solution of $(l - \lambda)g_\lambda = 0$ with initial conditions

$$\begin{aligned} g_\lambda(0) &= \sum_{j=0}^{k} (\lambda - \hat{\lambda})^j y^{[j]}(0) \\ g_\lambda{}'(0) &= \sum_{j=0}^{k} (\lambda - \hat{\lambda})^j y^{[j]\prime}(0). \end{aligned}$$

Straightforward computation yields

$$y^{[j]} = \frac{1}{j!} \left. \frac{\partial^j g_\lambda}{\partial \lambda^j} \right|_{\lambda = \hat{\lambda}}.$$

Setting

$$f_\lambda = \frac{w(z_1, ..., z_m, y^{[0]}, g_\lambda)}{w(z_1, ..., z_m, y^{[0]})},$$

we observe that $f_{\hat{\lambda}} = 0$ and

$$u^{[j-1]} = \frac{1}{j!} \left. \frac{\partial^j f_\lambda}{\partial \lambda^j} \right|_{\lambda = \hat{\lambda}}$$

for $j > 0$. From Theorem 3.2, we have

$$-f_\lambda{}'' + (q - 2W(z_1, ..., z_m, y^{[0]})')f_\lambda = \lambda f_\lambda,$$

which when differentiated j times with respect to λ gives

$$-\left(\frac{\partial^j f_\lambda}{\partial \lambda^j}\right)'' + (q - 2W(z_1, ..., z_m, y^{[0]})')\frac{\partial^j f_\lambda}{\partial \lambda^j} = \lambda \frac{\partial^j f_\lambda}{\partial \lambda^j} + j \frac{\partial^{j-1} f_\lambda}{\partial \lambda^{j-1}}.$$

We now divide by $j!$ and set $\lambda = \hat{\lambda}$ in the above equation to give the result. ■

4 Darboux Transformations

This section considers the simplest cases, where a (Darboux) transformation constructed from a single base function results in a Sturm-Liouville problem with constant type boundary conditions. In these cases, the "extra" eigenvalue noted in Section 1 appears on the zeroth branch $\mathcal{B}_0$ of the Prüfer graph. The transformation "removes" this eigenvalue, leaving one eigenvalue per branch.

We begin with the non-Dirichlet cases (labelled N_0 at the end of Section 2), when the base functions can be taken as eigenfunctions of (1.1)-(1.3).

It is convenient to use the notation $\Lambda = \{\lambda_j : j \geq 0\}$ and

$$\Lambda_n = \{\lambda_j : n \neq j \geq 0\}, \tag{4.1}$$

with the eigenvalues, λ_j, of (1.1)-(1.3) being labelled as in Theorem 2.1.

Theorem 4.1 *Let $\alpha > 0$ in (1.2) and assume that $\lambda_0 \leq \lambda_1$, both lying on the initial branch, $\mathcal{B}_0$, of the Prüfer graph. Then the Darboux transformation with base function y_0 (in the case of $\lambda_0 = \lambda_1$, we take as y_0 an eigenfunction for this eigenvalue and y_1 as $y_0^{[1]}$ the first associated function) produces a Sturm-Liouville problem with potential*

$$\hat{q} = q - 2Y_0',$$

boundary conditions

$$\begin{aligned} y(0) &= 0, \\ Y(1) &= -\frac{1}{a} - (a\lambda_0 + b), \end{aligned}$$

and spectrum Λ_0.

Proof: Theorems 3.2 and 3.4 show that the functions

$$u_j = \frac{w(y_0, y_j)}{y_0}$$

(and in the case of an eigenvalue of multiplicity 2, $u_1 = \frac{w(y_0, y_0^{[1]})}{y_0}$) are solutions of (1.1) with $\lambda = \lambda_j$, $j \geq 1$, and q replaced by $\hat{q}$.

As y_0 and y_j (and $y_0^{[1]}$ when considered) obey the same initial condition, which is λ-independent, it follows that $u_j(0) = 0$.

In the case of

$$u = \frac{w(y_0, y)}{y_0}$$

where y is a solution of (1.1) obeying (1.3) we have

$$U = W(y_0, y) - Y_0 = \frac{(\lambda_0 - \lambda)y_0 y}{w(y_0, y)} - Y_0,$$

for $\lambda \in \mathbb{C}$. When evaluated at 1 this gives

$$U(1) = -\frac{1}{a} - (a\lambda_0 + b).$$

Setting $y = y_j$ and $\lambda = \lambda_j$ in the above equation we obtain the required boundary conditions at 1 for the case when λ_j is a simple eigenvalue.

For $\lambda_0 = \lambda_1$ we take the λ derivatives of our expression for u and set $\lambda = \lambda_0$ giving

$$u_1 = \frac{w(y_0, \dot{y}|_{\lambda=\lambda_0})}{y_0} = \frac{w(y_0, y_0^{[1]})}{y_0}.$$

Consequently, via the equation $-y_0^{[1]''} + qy_0^{[1]} = \lambda_0 y_0^{[1]} + y_0$, we have

$$U_1 = W(y_0, y_0^{[1]}) - Y_0 = -\frac{y_0^2}{w(y_0, y_0^{[1]})} - Y_0,$$

but $y_0^{[1]}$ obeys the boundary condition

$$y_0^{[1]'}(1) = (a\lambda_0 + b)y_0^{[1]}(1) + ay_0(1)$$

and thus

$$U_1(1) = -\frac{1}{a} - (a\lambda_0 + b).$$

The transformed problem, which has a Dirichlet boundary condition at $x = 0$ and a non-Dirichlet constant type boundary condition at $x = 1$, has eigenvalues, $\mu_0 < \mu_1 < ...$, which take the asymptotic form

$$\mu_j = \pi^2 \left(j + \frac{1}{2}\right)^2 + O(1).$$

In addition we have shown that each of $\lambda_1, \lambda_2, ...$ is an (algebraically simple) eigenvalue of the transformed problem and from Theorem 2.1

$$\lambda_n = \left(n - \frac{1}{2}\right)^2 \pi^2 + O(1).$$

Thus $\lambda_{j+1} = \mu_j, j = 0, 1, ...,$ and Λ_0 constitutes the set of all eigenvalues for the transformed problem. ■

For Dirichlet cases we need to perturb the eigenfunctions to give the required base functions. The following theorem constructs one of the two perturbations needed in what follows.

Theorem 4.2 *If all eigenvalues of (1.1)-(1.3) are real and simple, then there are at least two eigenvalues with the same oscillation count, say* n*. In particular the second largest eigenvalue with oscillation count* n *is* λ_n *and the largest is* λ_{n+1}*. Then there exists* $K < \cot\alpha$ *so that, for each* β *with* $K < \cot\beta < \cot\alpha$ *(where a Dirichlet condition at* $x = 0$ *is interpreted as* $\cot\alpha = +\infty$*), there exists* μ *with* $\lambda_{n+1} > \mu > \lambda_n$ *so that the solution* z *of (1.1) with* $\lambda = \mu$ *and*

$$Z(0) = \cot\beta$$

has oscillation count n *and obeys the terminal condition*

$$Z(1) = a\mu + b.$$

Proof: The effect of decreasing $\cot\beta$ is to shift the (Prüfer) graph of $Z(1)$ to the left. We denote the nth branch of this graph by $\mathcal{B}_n(\beta)$.

If $n = 0$ then $a\lambda + b$ intersects $\mathcal{B}_n(\alpha) = \mathcal{B}_n$ twice, so under sufficiently small deformation of the Prüfer graph to the left, i.e. for $\cot\beta$ in some interval of the form $(K, \cot\alpha)$, $a\lambda + b$ still intersects $\mathcal{B}_n(\beta)$ with abscissae $\mu_1(\beta) < \mu_2(\beta)$, say, in the interval $(\lambda_n, \lambda_{n+1})$. The result follows if we take β as above and $\mu = \mu_1(\beta)$.

If $n > 0$ then the proof is the same except for the existence of a third intersection point with abscissa $\mu_0(\beta) < \mu_1(\beta)$. ■

Note 4.3 *In the case* $\lambda_{n-1} = \lambda_n < \lambda_{n+1}$ *(only possible if* $n \geq 1$*), the above theorem is still valid.*

We now consider, with the help of Theorem 4.2, the case of a Dirichlet boundary condition at 0 and two simple eigenvalues on the initial branch, $\mathcal{B}_0$, of the Prüfer graph.

Theorem 4.4 *Let* $\alpha = 0$ *in (1.2) and assume that* $\lambda_0 < \lambda_1$ *both lie on* $\mathcal{B}_0$*. Then the Darboux transformation with base function* z *as given in Theorem 4.2 produces a Sturm-Liouville problem with potential*

$$\hat{q} = q - 2Z'$$

and boundary conditions

$$\begin{aligned} Y(0) &= -\cot\beta = Z(0), \\ Y(1) &= -\frac{1}{a} - (a\mu + b), \end{aligned}$$

which is isospectral with (1.1)-(1.3).

Proof: Theorem 3.2 shows that for $\lambda = \lambda_j$, $j \geq 0$, and q replaced by $\hat{q}$ the functions

$$u_j = \frac{w(z, y_j)}{z}$$

are solutions of (1.1).

As in Theorem 4.1

$$U_j = \frac{(\mu - \lambda_j) z y_j}{w(z, y_j)} - Z.$$

The above expression evaluated at 0 and 1 (using the boundary conditions obeyed by z and y_j) gives

$$U_j(1) = -\frac{1}{a} - (a\mu + b)$$

and

$$U_j(0) = -Z(0).$$

Thus $\lambda_j, j = 0, 1, \ldots$ are eigenvalues of the transformed problem.

A comparison (as for Theorem 4.1) of the asymptotic form of the eigenvalues of the transformed problem and of λ_n as given in Theorem 2.1 (i) shows that Λ constitutes the set of all eigenvalues for the transformed problem. ■

Remark In the shorthand at the end of Section 2, we have covered all cases where (1.3) intersects $\mathcal{B}_0$, except for $D_0(2)$.

5 Double transformations

In this section we discuss all remaining cases with real eigenvalues. First we show that, when all eigenvalues are real and simple, a double Crum transformation converts (1.1)-(1.3) to a Sturm-Liouville boundary value problem with real constant boundary conditions.

Theorem 5.1 *Suppose that (1.1)-(1.3) has only real simple eigenvalues with $\lambda_{n-1} < \lambda_n < \lambda_{n+1}$ on $\mathcal{B}_n$. Let β, μ and z be as in Theorem 4.2. Then the Crum transformation with base functions z and y_n transforms (1.1)-(1.3) to the Sturm-Liouvlle problem*

$$-u'' + (q - 2W(y_n, z)')u = \lambda u \tag{5.1}$$

$$\begin{aligned} U(0) &= \cot\alpha + \frac{\lambda_n - \mu}{\cot\alpha - \cot\beta}, \quad \text{if } \alpha \neq 0 \\ u(0) &= 0, \quad \text{if } \alpha = 0 \end{aligned} \tag{5.2}$$

$$u(1) = 0 \tag{5.3}$$

which has as its spectrum Λ_n, (4.1).

Proof: Theorem 3.3 ensures that $w(y_n, z) \neq 0$. Throughout the proof $\lambda_j \in \Lambda_n$. Let

$$u_j = \frac{w(y_n, z, y_j)}{w(y_n, z)}.$$

It follows from Theorem 3.2 that u_j obeys (5.1) with $\lambda = \lambda_j$.

At 1 we have

$$w(y_n, z, y_j)(1) = \begin{vmatrix} y_n(1) & z(1) & y_j(1) \\ (a\lambda_n + b)y_n(1) & (a\mu + b)z(1) & (a\lambda_j + b)y_j(1) \\ -\lambda_n y_n(1) & -\mu z(1) & -\lambda_j y_j(1) \end{vmatrix} = 0,$$

which gives (5.3).

For $\alpha = 0$, let $y_j'(0) = 1 = {y_n}'(0)$ and $z(0) = 1$. Then

$$w(y_n, z, y_j)(0) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & \cot\beta & 1 \\ 0 & -\mu & 0 \end{vmatrix} = 0$$

and as $u_j'(0) \neq 0$ it follows that λ_j, are eigenvalues of (5.1)-(5.3) with eigenfunctions u_j. From [9]

$$\lambda_j = \pi^2 j^2 + O(1)$$

which is the asymptotic form of the eigenvalues of (5.1)-(5.3) since $\lambda_j \in \Lambda_n$. Thus Λ_n constitutes the set of eigenvalues of (5.1)-(5.3).

For $\alpha \neq 0$ assume that $y_n(0), y_j(0), z(0) = 1$. Then

$$\begin{aligned} w(y_n, z)(0) &= \cot\beta - \cot\alpha, \\ w(y_n, z, y_j)(0) &= \begin{vmatrix} 1 & 1 & 1 \\ \cot\alpha & \cot\beta & \cot\alpha \\ -\lambda_n & -\mu & -\lambda_j \end{vmatrix} \\ &= (\lambda_j - \lambda_n)(\cot\alpha - \cot\beta) \neq 0, \\ w'(y_n, z, y_j)(0) &= \begin{vmatrix} 1 & 1 & 1 \\ \cot\alpha & \cot\beta & \cot\alpha \\ -\lambda_n \cot\alpha & -\mu\cot\beta & -\lambda_j\cot\alpha \end{vmatrix} \\ &= (\lambda_j - \lambda_n)(\cot\alpha - \cot\beta)\cot\alpha. \end{aligned}$$

Thus u_j is not identically zero and obeys the boundary conditions (5.2) and (5.3), showing that the λ_j are eigenvalues of (5.1)-(5.3). From [9]

$$\lambda_j = \pi^2 \left(j - \frac{1}{2}\right)^2 + O(1)$$

which when compared with the asymptotic form for the eigenvalues of (5.1)-(5.3), shows that Λ_n is the set of all eigenvalues of (5.1)-(5.3). ■

For the case $\alpha > 0$ there is a mirror version of Theorem 4.2, stated below, and the note thereafter. The proof is omitted as it is similar to that of Theorem 4.2.

Theorem 5.2 *If $\alpha > 0$, all eigenvalues of (1.1)-(1.3) are real and $\lambda_n = \lambda_{n+1}$ is an algebraically double eigenvalue on $\mathcal{B}_n$, then either $n = 0$ or λ_{n-1} is on $\mathcal{B}_n$ if $n \geq 1$. Also there exists $K > \cot\alpha$ and μ with $\lambda_{n-1} < \mu < \lambda_n$ (where λ_{-1} is taken as $-\infty$), so that for each β with $K > \cot\beta > \cot\alpha$, the solutions z of (1.1) with $\lambda = \mu$ and*

$$Z(0) = \cot\beta$$

have oscillation count n and obey the terminal condition

$$Z(1) = a\mu + b.$$

Remark Theorem 5.2 also applies to eigenvalues of multiplicity 3, in which case $\mu > \lambda_{n+1}$.

The next result treats some cases with both single and double eigenvalues on $\mathcal{B}_k (k > 0)$, specifically $N_k(2,1), D_k(2,1)$ and $N_k(1,2)$ in the shorthand of Section 2.

Theorem 5.3 *Let β, μ and z be given by Theorem 4.2 in the case $\lambda_{n-1} = \lambda_n < \lambda_{n+1} \in \mathcal{B}_n$, and Theorem 5.2 in the case $\alpha > 0$, $\lambda_{n-1} < \lambda_n = \lambda_{n+1} \in \mathcal{B}_n$. Then the Crum transformation of (1.1)-(1.3) with base functions z and the eigenfunction y_n for the eigenvalue λ_n gives the boundary value problem (5.1)-(5.3) with spectrum Λ_n, (4.1).*

Proof: Let

$$u_j = \begin{cases} \frac{w(z,y_n,y_j)}{w(z,y_n)}, & \lambda_j \neq \lambda_n \\ \frac{w(z,y_n,y_n^{[1]})}{w(z,y_n)}, & j \neq n, \lambda_j = \lambda_n \end{cases}$$

where $y_n^{[1]}$ is the first associated function at $\lambda = \lambda_n$ corresponding to the eigenfunction $y_n = y_n^{[0]}$. For $\lambda_j \neq \lambda_n$ that u_j is an eigenfunction of (5.1)-(5.3) with eigenvalue λ_j, is proved exactly as in Theorem 5.1.

We now consider $j \neq n$ with $\lambda_j = \lambda_n$. From Theorem 3.4, u_j is a solution of (5.1) with $\lambda = \lambda_j$, so it remains only to show that u_j satisfies the boundary conditions (5.2)-(5.3).

As λ_n is an eigenvalue with Jordan chain of length two we have

$$y_n^{[1]\prime}(0)\sin\alpha = y_n^{[1]}(0)\cos\alpha$$

and

$$y_n^{[1]\prime}(1) = (a\lambda_n + b)y_n^{[1]}(1) + ay_n(1).$$

Thus at 1 we have

$$w(y_n, z, y_n^{[1]})(1) = \begin{vmatrix} y_n(1) & z(1) & y_n^{[1]}(1) \\ (a\lambda_n + b)y_n(1) & (a\mu + b)z(1) & (a\lambda_n + b)y_n^{[1]}(1) + ay_n(1) \\ (q - \lambda_n)y_n(1) & (q-\mu)z(1) & (q-\lambda_n)y_n^{[1]}(1) - y_n(1) \end{vmatrix} = 0,$$

thus giving $u_j(1) = 0$.

For $\alpha = 0$, let y_n and z be normalized by $y_n'(0) = 1$ and $z(0) = 1$ giving

$$w(y_n, z, y_n^{[1]})(0) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & \cot\beta & 0 \\ 0 & -\mu & 0 \end{vmatrix} = 0$$

and, as $u_j'(0) \neq 0$, it follows that λ_j is an eigenvalue of (5.1)-(5.3) with eigenfunction u_j.

For $\alpha \neq 0$ assume that y_n and z are normalized by $y_n(0) = 1 = z(0)$. Then

$$\begin{aligned} w(y_n, z, y_n^{[1]})(0) &= \begin{vmatrix} 1 & 1 & 0 \\ \cot\alpha & \cot\beta & 0 \\ -\lambda_n & -\mu & -1 \end{vmatrix} \\ &= \cot\alpha - \cot\beta \neq 0, \\ w'(y_n, z, y_n^{[1]})(0) &= \begin{vmatrix} 1 & 1 & 0 \\ \cot\alpha & \cot\beta & 0 \\ -\lambda_n \cot\alpha & -\mu\cot\beta & -\cot\alpha \end{vmatrix} \\ &= (\cot\alpha - \cot\beta)\cot\alpha. \end{aligned}$$

Thus u_j is not identically zero and obeys the boundary conditions (5.2) and (5.3), showing that λ_j is an eigenvalue of (5.1)-(5.3).

Appealing, as in Theorem 5.1, to the asymptotics for the eigenvalues of (5.1)-(5.3) and those given for (1.1)-(1.3) given in Theorem 2.1, we obtain that $\lambda_j, j \neq n$, consitute the spectrum of the transformed boundary value problem. ■

In the remaining cases with real spectrum, the perturbation results of Theorems 4.2 and 5.2 cannot be applied. Here, instead of using a Crum transformation to remove an eigenvalue, we use it to split double eigenvalues into two simple eigenvalues and triple eigenvalues into a double and a simple eigenvalue. The following lemma is a consequence of the fact that the asymptotes of the Prüfer graph move continuously to the left as $\cot\beta$ decreases, in the situation of Theorem 4.2 (even when there are multiple eigenvalues).

Lemma 5.4 *Let $\lambda_{n-1} \leq \lambda_n = \lambda_{n+1}$ (where λ_{n-1} is ignored if $n = 0$) lie on $\mathcal{B}_n$. Then there exists K such that whenever $K < \cot\beta < \cot\alpha$ we have*

$$\lambda_{n+1} < \lambda_n^D(\beta) \in \mathcal{B}_n \tag{5.4}$$

where $\lambda_n^D(\beta)$ denotes the nth eigenvalue of (1.1)-(1.3) with α replaced by β and (1.3) replaced by the Dirichlet condition $y(1) = 0$.

Now we are ready to discuss triple eigenvalues, and the remaining double eigenvalue Dirichlet cases, $D_0(2)$ and $D_k(1,2)$.

Theorem 5.5 *If $\alpha = 0$ suppose that $\lambda_n = \lambda_{n+1}$ is the largest eigenvalue on $\mathcal{B}_n$ while if $\alpha \neq 0$ suppose that $\lambda_{n-1} = \lambda_n = \lambda_{n+1}$. Let β and $\lambda_n^D(\beta)$ be as in Lemma 5.4 and z be a non-trivial solution of (1.1) with $\lambda = \lambda_n^D(\beta)$ and*

$$Z(0) = \cot\beta.$$

Then the Crum transformation with base functions z and $y_n = y_n^{[0]}$ transforms (1.1)-(1.3) to (1.1)-(1.2) with q replaced by $q - 2W[z, y_n]'$ and (1.3) replaced by

$$Y(1) = a\lambda + a(\lambda_n - \lambda_n^D(\beta)) + b.$$

The eigenvalues of the transformed boundary value problem are $\lambda_j, j \neq n$, together with $\lambda_n^D(\beta)$ (which replaces λ_n).

Proof: Theorem 3.3 ensures that $w(z, y_n)$ does not vanish on $[0, 1]$. The proof of the theorem proceeds like the proofs of Theorems 5.1 and 5.5. Note that $\lambda_n^D(\beta) = \mu$ is an eigenvalue of the transformed problem with eigenfunction

$$e_\mu = \frac{y_n}{w(z, y_n)}.$$

This follows from Theorem 3.2 and is a routine calculation to check the boundary conditions. ■

The net result of the above theorem is that, while the Crum transformation has not directly given constant boundary conditions, it has produced a "lower multiplicity" problem to which the process can be applied again, leading to constant boundary conditions via Theorems 4.4, 5.1 and 5.3. Specifically, $D_k(3), N_k(3), D_0(2)$ and $D_k(1,2)$ transform to $D_k(2,1), N_k(2,1), D_0(1,1)$ and $D_k(1,1,1)$, respectively. This follows from (5.4) and the fact that the above transformation preserves eigenvalue oscillation count. To see this, consider (1.1)-(1.2) with (1.3) replaced by a Dirichlet condition. The transformation used in Theorem 5.5 is isospectral for this problem and provides the same transformed problem as given in Theorem 5.5 but with a Dirichlet condition at 1. Thus the projection onto the λ-axis of the branches $\mathcal{B}_k$ for the original problem and for the transformed problem are identical.

6 Complex Spectrum

The cases remaining for study involve a complex conjugate pair of non-real eigenvalues.

Lemma 6.1 *Suppose that (1.1)-(1.3) has a conjugate pair of non-real eigenvalues* $\lambda = \rho + i\sigma$ *and* $\bar{\lambda}$ *with eigenfunctions* f *and* $\bar{f}$*. Then* f *has no zeros in* $(0, 1]$.

Proof: As f and $\bar{f}$ obey the same initial condition at 0, Lagrange's formula gives

$$w(f, \bar{f})(x) = \int_0^x [f\bar{f}'' - f''\bar{f}] = 2i\sigma \int_0^x |f|^2. \tag{6.1}$$

Were $f(x_0) = 0$ for some $x_0 \in (0, 1]$, this would yield $\int_0^{x_0} |f|^2 = 0$, a contradiction. ■

Remark 6.2 In the case $\alpha \neq 0$, we may assume $f(0) = 1$ so there exist $k > c > 0$ with $k \geq |f(x)| \geq c$ for all $x \in [0, 1]$, i.e., we can write $f(x) = r(x)e^{i\Theta(x)}$ where r and Θ are continuous, $\Theta(0) = 0$ and $k \geq r(x) \geq c$ for all $x \in [0, 1]$. In the case $\alpha = 0$, we have $f(0) = 0$ and may assume $f'(0) = 1$. Thus $f(x)/x$ has continuous extension, $\hat{f}(x)$, to $[0, 1]$ with $\hat{f}(0) = 1$. Hence setting $|\hat{f}(x)| = g(x)$ we have $k_1 > c_1 > 0$ with $c_1 \leq g(x) \leq k_1$ for all $x \in [0, 1]$ and a unique continuous real valued function Θ with $\Theta(0) = 0$ and $\hat{f}(x) = g(x)e^{i\Theta(x)}$. In particular $f = re^{i\Theta}$ where $r(x) = xg(x)$ and $r'(0) = g(0) = 1$.

Theorem 6.3 *Suppose* $\lambda = \rho + i\sigma$, $\sigma > 0$, *is a non-real eigenvalue for (1.1)-(1.3) with eigenfunction* $f(x) = r(x)e^{i\Theta(x)}$ *where* $\Theta(0) = 0$ *and either* $r(0) = 1$ *if* $\alpha > 0$ *or* $r'(0) = 1$ *if* $\alpha = 0$*. There is* $\hat{\beta} \in (0, \pi)$ *such that for each* μ *sufficiently negative and for each* $\beta_1 \in (\hat{\beta}, \pi)$*, there are* $\beta_0 \in (\alpha, \pi)$ *and* z *having no zeros in* $[0, 1]$*, such that*

$$-z'' + qz = \mu z, \qquad Z(0) = \cot \beta_0, \qquad Z(1) = \cot \beta_1,$$

and $R(x) > Z(x)$ *for all* $x \in [0, 1]$ *(for all* $x \in (0, 1]$ *if* $\alpha = 0$*).*

Proof: We give details for the case $\alpha > 0$. The adjustments needed for $\alpha = 0$ are straightforward in the light of Remark 6.2.

We note from Lemma 6.1 that $r(x) > 0$ for all $x \in [0, 1]$. With $\Phi = \Theta'$ we have

$$F = R + i\Phi$$

and so $R(0) = \cot \alpha, \Phi(0) = 0$.

Now (1.1) is

$$-(re^{i\Theta})'' + qre^{i\Theta} = (\rho + i\sigma)re^{i\Theta},$$

and by equating real and imaginary parts, we have

$$-r'' + (q + \Phi^2)r = \rho r \tag{6.2}$$

$$r\Phi' + 2r'\Phi + \sigma r = 0. \tag{6.3}$$

From (6.1) and the equation

$$w(f, \bar{f}) = -2i\Theta' r^2$$

we obtain

$$\Phi(x) = -\frac{\sigma}{r^2(x)} \int_0^x r^2(s)\, ds.$$

This along with Remark 6.2 shows that Φ has a continuous extension to $[0,1]$ with $\Phi(0) = 0$ and negative on $(0,1]$. Hence there is a constant $\kappa > 0$ such that $-\kappa \leq \Phi(x) \leq 0$ for all $x \in [0,1]$.

We now consider (6.2) as a Sturm-Liouville equation with initial condition $\frac{r'}{r}(0) = \cot\alpha$. The corresponding Prüfer angle θ_r, say, satisfies

$$\theta_r{}'(x) = \cos^2\theta_r(x) + (\rho - q(x) - \Phi^2(x))\sin^2\theta_r(x), \qquad \theta_r(0) = \alpha,$$

and since $r \neq 0$ on $[0,1]$, it follows that $0 < \theta_r(1) < \pi$. Select $\beta_1 \in (\theta_r(1), \pi)$ and let $\hat{\mu} > \kappa^2$ be arbitrary. The Prüfer equation

$$\theta' = \cos^2\theta + (\rho - q - \hat{\mu})\sin^2\theta, \qquad \theta(1) = \beta_1$$

has a unique solution $\hat{\theta}$, say, and the Comparison Theorem shows that $\hat{\theta}(x) > \theta_r(x)$ for all $x \in [0,1]$. Thus $\hat{\theta}(0) > \alpha$ and moreover $\hat{\theta}(0) < \pi$ since $\hat{\theta}$ can only increase through multiples of π. We take $\beta_0 = \hat{\theta}(0)$. The upshot is that there is a function z on $[0,1]$ for which $Z(x) = \cot\hat{\theta}(x)$ satisfying the demands of the lemma with $\mu = \rho - \hat{\mu}$. ■

The non-Dirichlet case ($\alpha > 0$). We list the spectrum as $\lambda_0,\ \bar{\lambda}_0,\ \lambda_1,\ \lambda_2,\ \cdots$ with corresponding eigenfunctions as $f,\ \bar{f},\ y_1,\ y_2,\ \cdots$. We construct z as in Lemma 6.3 and calculate

$$w\left(z, f, \bar{f}\right) = -\begin{vmatrix} z & f & \bar{f} \\ z' & f' & \bar{f}' \\ \mu z & \lambda_0 f & \bar{\lambda}_0 \bar{f} \end{vmatrix}$$

which after some manipulation simplifies to

$$2iz|f|^2((\mu - \rho)\Phi + \sigma(R - Z)).$$

Here, as above, $\lambda_0 = \rho + i\sigma, \sigma > 0$. Lemma 6.3 now shows that $w\left(z, f, \bar{f}\right)$ does not vanish on $[0,1]$.

This leads us to a Crum transformation with three base functions z, f and $\bar{f}$ generating a new potential $\hat{q} = q - 2W(z, f, \bar{f})'$. The eigenfunctions

$$e_n = \frac{w\left(z, f, \bar{f}, y_n\right)}{w\left(z, f, \bar{f}\right)}, \qquad n \geq 1$$

for eigenvalues λ_n, $n \geq 1$ satisfy

$$e_n(0) = 0, \qquad E_n(1) = -\frac{1}{a} - W(z, f, \bar{f})(1).$$

These calculations are easily performed. It is important to note that μ is also an eigenvalue for the transformed problem with eigenfunction

$$e_\mu = \frac{w\left(f, \bar{f}\right)}{w\left(z, f, \bar{f}\right)}.$$

Indeed, Theorem 3.2 verifies that e_μ obeys the transformed differential equation, while tedious but routine calculations give the new boundary conditions.

We can summarise this discussion with the following theorem.

Theorem 6.4 *If (1.1)-(1.3) has $\alpha > 0$ and non-real eigenvalues then there is a Crum transformation with three base functions (two of which are eigenfunctions for the conjugate pair of non-real eigenvalues) transforming (1.1)-(1.3) to a problem with Dirichlet condition at 0 and non-Dirichlet (constant) boundary condition at 1 and the same spectrum as (1.1)-(1.3) but with the non-real eigenvalues replaced by one real eigenvalue μ below the least real eigenvalue of the initial problem.*

The net result is that the transformed problem has constant boundary conditions, the initial condition being Dirichlet, and spectrum μ, λ_1, λ_2, $\cdots$.

The Dirichlet case. Modifications must be made to the above method when $\alpha = 0$. We select μ large and negative and construct z as before (along with β_0 and β_1). Then with $\nu < \mu$ and $\tilde{\beta}_1 > \beta_1$, we repeat the construction to obtain another nonvanishing function v with

$$-v'' + qv = \nu v, \quad V(0) = \tilde{\beta}_0, \quad V(1) = \tilde{\beta}_1, \quad \tilde{\beta}_1 > \beta_1, \quad \tilde{\beta}_0 > \beta_0$$

and evidently,

$$R(x) > Z(x) > V(x), \qquad x \in (0, 1].$$

Then we can verify

$$w\left(v, z, f, \bar{f}\right)(0) = 2i\sigma v(0) z(0) \left|f'(0)\right|^2 (\mu - \nu)$$

and

$$\begin{aligned} \frac{w\left(v, z, f, \bar{f}\right)}{2ivz|f|^2} &= (V - Z)\Phi\sigma^2 + (\mu - \nu)\sigma\Phi^2 \\ &\quad + (V - Z)(\rho - \nu)(\rho - \mu)\Phi + (\mu - \nu)(R - V)(R - Z)\sigma \end{aligned}$$

on $(0, 1]$. Since $\Phi < 0$ on $(0, 1]$ and $\rho > \mu > \nu$ we see that $w(v, z, f, \bar{f})$ does not vanish on $[0, 1]$.

This leads to the use of a Crum transformation with the four base functions v, z, f, $\bar{f}$ producing a new problem with eigenvalues $\nu < \mu < \lambda_1 < \lambda_2 < \cdots$, eigenfunctions

$$e_\nu = \frac{w(z, f, \bar{f})}{w(v, z, f, \bar{f})}, \quad e_\mu = \frac{w(v, f, \bar{f})}{w(v, z, f, \bar{f})}, \quad e_n = \frac{w(v, z, f, \bar{f}, y_n)}{w(v, z, f, \bar{f})}, \ n \geq 1,$$

and boundary conditions

$$y(0) = 0, \quad Y(1) = c\lambda + d, \tag{6.4}$$

for some real constants c and d. As before, Theorem 3.2 verifies that e_ν, e_μ and e_n are solutions of the transformed differential equation, and the boundary conditions (6.4) follow from a routine, but tedious, calculation.

In brief, we have deduced the following theorem.

Theorem 6.5 *If (1.1)-(1.3) has $\alpha = 0$ and non-real eigenvalues then there is a Crum transformation with four base functions (two of which are eigenfunctions for the conjugate pair of non-real eigenvalues) which transforms (1.1)-(1.3) to a problem with boundary conditions (6.4) and with the same spectrum as (1.1)-(1.3) but with the non-real eigenvalues replaced by two (distinct) real eigenvalues below the least real eigenvalue of the initial problem.*

Finally we note that although $c < 0$ (since the new eigenfunctions e_ν, e_μ both have no internal zeros in $(0, 1)$), the transformed problem is of type $D_0(1, 1)$, to which Theorem 4.4 can be applied.

References

[1] V. É. Adler, A modification of Crum's method, *Theoret. and Math. Phys.* **101** (1994), 1381-1386.

[2] F. V. Atkinson, *Discrete and Continuous Boundary Value Problems,* Academic Press, New York, 1964.

[3] A. I. Benedek, R. Panzone, On Sturm-Liouville problems with the square-root of the eigenvalue parameter contained in the boundary condition, *Notas de Algebra y Analisis*, **10** (1981), 1-59.

[4] P. A. Binding, P. J. Browne, Applications of two parameter eigencurves to Sturm-Liouville problems with eigenparameter-dependent boundary conditions, *Proc. Roy. Soc. Edinburgh*, **125A** (1995), 1205-1218.

[5] P. A. Binding, P. J. Browne, K. Seddighi, Sturm Liouville problems with eigenparameter dependent boundary conditions, *Proc. Edin. Math. Soc.*, **37** (1993), 57-72.

[6] P. A. Binding, P. J. Browne, B. A. Watson, Transformations between Sturm-Liouville problems with eigenvalue dependent and independent boundary conditions, *Bull. London Math. Soc.*, **33** (2001), 749-757.

[7] P. A. Binding, P. J. Browne, B. A. Watson, Sturm Liouville problems with boundary conditions rationally dependent on the eigenparameter, I, *Proc. Edin. Math. Soc.*, **45** (2002), 631-645.

[8] P. A. Binding, P. J. Browne, B. A. Watson, Sturm Liouville problems with boundary conditions rationally dependent on the eigenparameter, II, *Proc. Edin. Math. Soc.*, **148** (2002), 147-168.

[9] P. A. Binding, P. J. Browne, B. A. Watson, Equivalence of inverse Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, *J. Math. Anal. Applic.*, to appear.

[10] L. Collatz, *Eigenwertaufgaben mit technischen Anwendungen*, Akademische Verlag, Leipzig, 1963.

[11] M. M. Crum, Associated Sturm-Liouville Systems, *Quart. J. Math. Oxford (2)* **6** (1955), 121-127.

[12] A. Dijksma, H. Langer, Operator theory and ordinary differential operators, *Fields Inst. Mono.* **3** (1996), 75-139.

[13] C. T. Fulton, Two-point boundary value problems with eigenparameter contained in the the boundary conditions, *Proc. Roy. Soc. Edinburgh*, **77A** (1977), 293-308.

[14] E. L. Ince, *Ordinary Differential Equations,* Dover, 1956.

[15] M. A. Naimark, *Linear Differential Operators, Vol I*, Frederick Ungar, New York, 1967.

[16] E. M. Russakovskii, Operator treatment of boundary problems with spectral parameters entering via polynomials in the boundary conditions, *Functional Anal. Appl.*, **9** (1975), 358-359.

[17] A. V. Štraus, On spectral functions of differential operators, *Izvest. Akad. Nauk SSSR Ser. Mat.,* **19** (1955), 201-220.

[18] J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary conditions, *Math. Z.,* **133** (1973), 301-312.